40. Let $I \triangleleft R$.

(a) Define a *coset* as $r + I = \{r + i \mid i \in I\}$. Verify that two such cosets are either equal, or disjoint.

(b) Define an addition and a multiplication for the cosets by $(r + I) + (s + I) = (r + s) + I$ and $(r + I)(s + I) = rs + I$. Show that we get a ring. This ring is the *factor ring* R/I.

41. To which well-known rings are isomorphic the factor rings?

(a) $\mathbf{Z}/(m)$; (b) $\mathbf{Z}_{30}/(5)$; (c) $\mathbf{Z}_{30}/(8)$; (d) $\mathbf{Z}[x]/(x)$; (e) $\mathbf{Z}[x]/(2)$; (f) $\mathbf{Z}[x]/(2, x^2 + x + 1)$.

42. Which of the factor rings are fields?

(a) $\mathbf{Q}[x]/(x - 2)$; (b) $\mathbf{Q}[x]/(x^2 - 2)$; (c) $\mathbf{R}[x]/(x^2 - 2)$; (d) $\mathbf{Q}[x]/(x^2 + 1)$;
(e) $\mathbf{R}[x]/(x^2 + 1)$; (f) $\mathbf{C}[x]/(x^2 + 1)$; (g) $\mathbf{Z}_2[x]/(x^2 + 1)$; (h) $\mathbf{Z}_3[x]/(x^2 + 1)$.

43. Let $I \triangleleft R$, $R \neq I$. True or false:

(a1) If R is commutative, then so is R/I. (a2) If R/I is commutative, then so is R.
(b1) If R is zero-divisor free, then so is R/I. (b2) If R/I is zero-divisor free, then so is R.
(c1) If R is a field, then so is R/I. (c2) If R/I is a field, then so is R.
(d1) If R has an identity, then so does R/I. (d2) If R/I has an identity, then so does R.

44. Consider the ring $\mathcal{P}(X)$ in Problem 24e, i.e. the set of all subsets of X where addition is the symmetric difference and multiplication is the intersection.

(a) Characterize the principal ideals.

(b) Show that if X is a finite set, then all ideals in $\mathcal{P}(X)$ are principal ideals.

(c) Verify that if X is an infinite set, then its finite subsets form an ideal which cannot be generated by finitely many elements.

(d) Prove that for any $Y \subseteq X$, the factor ring $\mathcal{P}(X)/(Y)$ is isomorphic to $\mathcal{P}(X \setminus Y)$.

45. A ring *homomorphism* is a map from a ring R to a ring S which preserves the operations, i.e. $\varphi : R \to S$ where $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$ and $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$.

(a) Verify $\varphi(0) = 0$ and $\varphi(-a) = -\varphi(a)$.

(b) The *kernel* is the set of elements in R mapped into 0_S, i.e. $\operatorname{Ker} \varphi = \{r \in R \mid \varphi(r) = 0\}$. Prove $\operatorname{Ker} \varphi \triangleleft R$.

(c) The *image* is $\operatorname{Im} \varphi = \{\varphi(r) \mid r \in R\}$. Show $\operatorname{Im} \varphi \leq S$.

46. Which maps are ring homomorphisms? For the homomorphisms, determine the kernel and the image.

(a) $\varphi : \mathbf{Z} \to \mathbf{Z}$ where $\varphi(k) = 2k$; (b) $\varphi : \mathbf{R}[x] \to \mathbf{R}$ where $\varphi(f) = f(\pi)$;
(c) $\varphi : \mathbf{R}[x] \to \mathbf{R}[x]$ where $\varphi(f) = f'$; (d) $\varphi : \mathbf{C} \to \mathbf{C}$ where $\varphi(z) = \overline{z}$.

47. *Isomorphism* is a bijective homomorphism. If $\varphi : R \to S$ is an isomorphism, then R and S are *isomorphic*(=are "of the same form"), i.e. they are essentially the same, just the elements and operations bear different names.

(a) Prove that φ is an isomorphism $\iff$ ($\operatorname{Ker} \varphi = 0$ and $\operatorname{Im} \varphi = S$).

(b) Which rings are isomorphic: (b1) $\mathbf{Q}$ and $\mathbf{R}$; (b2) even numbers and multiples of 3; (b3) $\mathbf{Z}_9$ and the even residues in $\mathbf{Z}_{18}$; (b4) $\mathbf{Z}_{10}$ and the even residues in $\mathbf{Z}_{20}$.

48. Investigate the analog of Problem 43 for R and $\operatorname{Im} \varphi$ where $\varphi : R \to S$ is a non-zero homomorphism.

49. Prove the following basic propositions:

(a) *Homomorphism theorem*: If $\varphi : R \to S$ is a homomorphism, then $\operatorname{Im} \varphi \cong R/\operatorname{Ker} \varphi$.

(b) *Natural homomorphism*: If $I \triangleleft R$, then $\psi : R \to R/I$ where $\psi(r) = r + I$ is a homomorphism with $\operatorname{Ker} \psi = I$ and $\operatorname{Im} \psi = R/I$.