

67. Which positive integers can be written as the difference of two squares and in how many ways?

To solve the analogous problem for sums instead of differences, we introduce the *Gaussian integers*: $\alpha = a + bi$, where a and b are integers. The Gaussian integers form an integral domain G , divisibility, unit, gcd, irreducible, and prime are defined as usual. We show that G is a Euclidean ring, hence it is (a PID, and so) a UFD. Therefore, the irreducible elements are the same as the primes, and we shall use the (shorter) name Gaussian primes. Also, we shall characterize all Gaussian primes. An important tool is the *norm*: $N(\alpha) = a^2 + b^2 = |\alpha|^2 = \alpha \cdot \bar{\alpha}$.

68. Prove:

- (a) $\alpha \mid N(\alpha)$; (b) $N(\alpha\beta) = N(\alpha)N(\beta)$; (c) $\alpha \mid \gamma \Rightarrow N(\alpha) \mid N(\gamma)$, but the converse is false;
 (d) $N(\alpha) = 0 \iff \alpha = 0$; (e) $N(\alpha) = 1 \iff \alpha$ is a unit $\iff \alpha = \pm 1, \pm i$.

69. Which Gaussian integers are divisible by $1 + i$?

70. Prove that G is a Euclidean ring with the norm as a Euclidean function: To every $\beta \neq 0, \alpha \in G$ there exist $\gamma, \varrho \in G$ satisfying $\alpha = \beta\gamma + \varrho$ and $N(\varrho) < N(\beta)$.

71. Verify: (a) $\alpha \mid \gamma \iff \bar{\alpha} \mid \bar{\gamma}$; (b) α is a Gaussian prime iff $\bar{\alpha}$ is a Gaussian prime.

72. Let $p > 2$ be a prime. Prove that the congruence $x^2 \equiv -1$ is solvable iff $p \equiv 1 \pmod{4}$.

73. We characterize all Gaussian primes π :

- (a) Every π divides exactly one positive prime p ;
 (b) Every p is either a Gaussian prime, or the product of two conjugate Gaussian primes;
 (c) $2 = (1 + i)(1 - i) = -i(1 + i)^2$ provides $\pi = 1 + i$ (and its associates);
 (d) Each prime $p \equiv -1 \pmod{4}$ is a Gaussian prime;
 (e) Each prime $p \equiv 1 \pmod{4}$ is the product of two conjugate, non-associate Gaussian primes:
 $p = \pi \cdot \bar{\pi}$.

74. Factor (a) $35000i$; (b) $270 + 2610i$; (c) $86 + 162i$ into the product of Gaussian primes.

75. True or false ($\alpha = a + bi$):

- (a) If α and β are coprime, then also $N(\alpha)$ and $N(\beta)$ are coprime.
 (b) If $N(\alpha)$ and $N(\beta)$ are coprime, then also α and β are coprime.
 (c) If a and b are coprime, then also α and $\bar{\alpha}$ are coprime.
 (d) If α and $\bar{\alpha}$ are coprime, then also a and b are coprime.
 (e) If α is a Gaussian prime, then $N(\alpha)$ is a prime number.
 (f) If $N(\alpha)$ is a prime number, then α is a Gaussian prime.
 (g) If α is the cube of a Gaussian integer, then $N(\alpha)$ is the cube of a non-negative integer.
 (h) If $N(\alpha)$ is the cube of a non-negative integer, then α is the cube of a Gaussian integer.

***76.** Which positive integers can be represented and in how many ways as the sum of squares of
 (a) two integers; (b) two *coprime* integers?

77. Determine the largest r such that there exist infinitely many sequences of r consecutive integers each being the sum or difference of two squares.

***78.** How many representations has a positive integer as the sum of two squares, in average? In a precise formulation, we ask about the approximate behavior of the mean value function

$$\frac{r(1) + r(2) + \dots + r(n)}{n}$$

for “large” values of n , where $r(n)$ denotes the number of integer solutions of the equation $x^2 + y^2 = n$.

***79.** Find all integer solutions of the equation $x^2 + 4 = y^3$.